Optimal Simultaneous Structural and Control Design of Maneuvering Flexible Spacecraft

Arthur L. Hale,* Ronald J. Lisowski,† and Wayne E. Dahl‡
University of Illinois at Urbana-Champaign, Urbana, Illinois

An optimization problem for maneuvering flexible spacecraft is discussed wherein both structural parameters and active control forces are to be determined so that a specific cost functional is minimized. The problem is an application of the general theory of optimal control of parametric systems. For simplicity, only maneuvers from a specified initial state to a specified final state in a specified time interval are considered. Numerical examples are presented for single-axis slew maneuvers of a symmetric four-boom flexible structure. The mass and stiffness distributions of the booms are determined as part of the optimization problem.

Introduction

TRANSPORTING massive objects into Earth orbit currently costs a great deal. As a result, spacecraft designers must strive to minimize a large structure's mass. When their efforts lead to a highly flexible structure, the shape must somehow be controlled actively. Many recent papers[1-14] address the problem of designing and implementing active control sytems for large flexible structures. The problem is complicated by the very small structural and/or viscous damping that is characteristic of structures in the space environment. Both linear vibration control[4-10] and maneuvers[11-14] have been considered. The prevailing attitude is that the control system is designed for a given structure, i.e., its designer has little or no input into the structural design.

This paper (a combination of Refs. 15 and 16) explores designing flexible structures with the active control system explicitly in mind. The discussion is restricted to maneuvering structures, both for simplicity and because in many cases optimal solutions exist. A maneuver is considered to be the transition from a known initial state (possibly a quiescent state) to a known final state (also possibly a quiescent state) in a known finite time interval. The premise is that maneuvering a very stiff structure, and, thus, one that is both massive and expensive to transport, requires considerable control energy because of its inertia. On the other hand, the same maneuver of a very low mass and, thus, very flexible structure can also require considerable control energy because of flexure. Because transportation costs rule out extremely massive designs, practical constraints on control energy and operational limitations on maneuvering time must be weighed against the transportation costs when designing a flexible structure.

First, mission objective must be examined closely. One vital and tacit assumption is that some mass is absolutely essential to accomplish a mission. A structure composed only of the essential mass possesses the minimum of mass permitted for the mission. The essential structure can be a collection of discrete masses or a continuum. For example, an essential structure for an antenna might consist of a very thin membrane that is free at all its edges.

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*Assistant Professor; joint appointment to Civil Engineering Department. Member AIAA.

†Major USAF, presently Assistant Professor and Deputy for Labs and Research, Department of Astronautics, USAF Academy, Colo. Member AIAA.

‡Graduate Research Assistant.

The mechanism that leads to increasing control costs for very flexible structures must also be specifically identified. The structure must be controllable. A control force on each mass of a system of disjoint masses is sufficient to allow control of the entire system without added structural components. However, if actuators are not available for some masses, structural components must connect these masses to others that do have control forces. In the case of a distributed essential structure, unbounded spatially distributed control forces are sufficient to control the structure in any desired way. However, if discrete control forces, i.e., discrete actuators, must be used on a distributed essential structure, the control cost can possibly be decreased by stiffening the structure. In a sense, the added structural stiffness, accomplished at the expense of increasing the structure's total mass, distributes the effect of discrete control forces over the essential structure.

One objective of this paper is to break ground in developing an analytical approach to the optimal structural design/control design problem. Antoher objective is to investigate the solution of the resulting nonlinear optimization problem. Linear quadratic cost control theory is used to design an optimal control for known structural parameters (e.g., mass density and stiffness). In contrast to Refs. 11-14, however, a weighted total mass term is added to the cost functional and the structural parameters are varied to find the minimum total cost. For simplicity, the designed structure is considered to be governed by a system of linear ordinary differential equations. In Ref. 15, translational maneuvers of an essential structure consisting of three discrete masses were presented. Herein the numerical examples of Ref. 16 concerning single axis slew maneuvers of a symmetric four boom flexible structure are presented.

Optimal Control of Parametric Structures

In this section a generic class of optimization problems is defined specifically for maneuvers of flexible structures. Because the emphasis is on structures, the formulation commences with a system of second-order differential equations. The equations constitute a spatially discrete mathematical model of the structure, which can contain discrete elements as well as spatially distributed ones. A complex structure made of many distributed components actually should be represented by a system of partial differential equations. For such structures the discrete equations considered here are the result of spatially discretizing the partial differential equations. As is customary in structural analysis, the vectors of generalized coordinates u and generalized forces f will be referred to as displacement and force vectors, respectively, although their entries may not correspond directly with any physical quantities. An appropriate discretization for the structural design parameters is also assumed.

Problem Statement

A control force $f(t)$ is desired that, in the time interval $[0,t_f]$, maneuvers the flexible structure described by the ordinary differential equations

$$[M(\xi)]\ddot{u}(t)+[C(\xi)]\dot{u}(t)+[K(\xi)]u(t)=[G(\xi)]f(t) \quad (1)$$

from the specified initial position $u(0)$ and initial velocity $v(0)$ to the final position $u(t_f)\in U$ and final velocity $v(t_f)\in V$, where U and V are sets containing the final position and velocity. In Eq. (1), $[M(\xi)]$, $[C(\xi)]$, and $[K(\xi)]$ are the symmetric $N\times N$-dimensional mass, damping, and stiffness matrices, respectively, depending on the N_d-dimensional vector of structural parameters ξ. Also, $[G(\xi)]$ is an $N\times N_c$ input matrix $(N_c\leq N)$ reflecting the spatial distribution of actuators that apply the N_c control forces. If ξ is considered known, Eq. (1) is linear.

Simultaneously with finding a control force, it is desired to design the structure, i.e., to determine values of the structural parameters. A control force and a set of structural parameters are desired that minimize the cost functional

$$J(f,\xi)=\alpha P(\xi)+\int_0^{t_f}\frac{1}{2}(f^T[R]f+\dot{u}^T[Q_1]\dot{u}+u^T[Q_0]u)\mathrm{d}t \quad (2)$$

where $[R]$ is an $N_c\times N_c$, and $[Q_1]$, $[Q_0]$ are $N\times N$ symmetric weighting matices. $[R]$ is positive definite, $[Q_1]$, $[Q_0]$ are positive semidefinite, and $\alpha P(\xi)$ represents a nonnegative cost depending on the structural parameters only (α is a nonnegative number). The structural parameters are constrained to be nonnegative by the N_d constraints

$$\xi_i\geq 0, \qquad i=1,2,\ldots,N_d \quad (3)$$

where ξ_i is the ith entry in ξ. Other explicit constraints, such as upper bounds on structural parameters, and upper and lower bounds on control forces, on displacements, or on velocities, are not imposed for simplicity.

When the interest is in mass minimization, as it is here, the function $\alpha P(\xi)$ represents a weighted mass term. The structural parameters ξ_i are chosen to be proportional to physical quantities such as cross sectional areas of bar elements, widths of beam elements, and thicknesses of membrane or plate elements. Realistically, the parameters cannot be negative, a criterion reflected in the constraints.[3]

Significance of $\alpha P(\xi)$

The problem stated above is a scalar optimization problem, i.e., $J(f,\xi)$ is scalar-valued. The scalar problem can be thought of as arising from a more general optimization problem that has a vector-valued cost.[17,18] For example, consider the simple case in which the vector cost has two entries: the first entry is proportional to the mass function $P(\xi)$, and the second entry is the integral term in Eq. (2), the maneuver cost. Ideally, one would like to minimize both entries of the cost vector simultaneously. Such a utopian[17,18] solution, however, rarely exists. Here, if the parameters are defined properly the mass function has a global minimum at $\xi=0$. On the other hand, except for special cases occurring when spatially distributed control forces are available, the maneuver cost is not a minimum at $\xi=\mathbf{0}$. In fact, in many problems the maneuver cost increases without bound as ξ approches $\mathbf{0}$ due to a loss of controllability. The probem, then, is to decide the relative importance of minimizing the mass vs minimizing the maneuver cost. Corresponding to the entire range of such decisions is a range of *Pareto-optimal*[17,18] solutions. A control-parameter pair that globally minimizes Eq. (2), a weighted sum of the vector entries, can be shown[17] to be Pareto-optimal for any value of α $(0\leq\alpha\leq\infty)$. The presence of $\alpha P(\xi)$ in Eq. (2) confines the solution to a region near $\xi=\mathbf{0}$ for which the maneuver cost is not too much greater than its globally minimum value.

The presence of $\alpha P(\xi)$ in Eq. (2) also is beneficial for purely computational reasons. Whereas the maneuver cost can be a complicated function of ξ which is only locally convex, the mass $P(\xi)$ is a well behaved convex function. Hence, the sum (2) for α large enough has a pronounced global minimum. If the interest is in a structural design that minimizes the maneuver cost, one can first solve the well behaved problem with $\alpha\neq 0$ and use the solution as a starting point for iteratively finding a solution to the problem with $\alpha=0$. The starting point is, it is hoped, in a locally convex region around the global minimum of the maneuver cost.

Necessary Conditions

Necessary conditions for an extremal control-parameter pair can be derived by the calculus of variations[19] or by geometric methods.[17] The functional $J(f,\xi)$ is to be minimized subject to the differential equation constraints (1) and the inequality constraints (3). Because ξ and $P(\xi)$ are not functions of time, the necessary conditions are separated into 1) those for optimal control of a structure with given parameters ξ, and 2) those for optimal structural parameters.

In deriving the necessary conditions, an augmented cost functional $J_a(f,\xi)$ is first formed by introducing the N-dimensional vector of auxiliary coordinates defined as $v=\dot{u}$, and the N-dimensional vectors w and u^* of Lagrange multipliers and adjoint displacements, respectively. Explicitly, the augmented functional is

$$J_a(f,\xi)=\alpha P(\xi)+\int_0^{t_f}\frac{1}{2}(f^T[R]f+v^T[Q_1]v+u^T[Q_0]u)\,\mathrm{d}t$$
$$+\int_0^{t_f}\{w^T[S](v-\dot{u})+u^{*T}([M(\xi)]\dot{v}$$
$$+[C(\xi)]v+[K(\xi)]u-[G(\xi)]f)\}\mathrm{d}t \quad (4)$$

Note that $[S]$ is any positive definite symmetric matrix; it is usually taken as the identity matrix $[1]$.

Necessary Conditions for Optimal Control (Parameters Given)

If the final time t_f is fixed and ξ is specified, taking the first variation of J_a yields the $4N+N_c$ necessary conditions for an optimal control

$$f=[R]^{-1}[G(\xi)]^T u^* \quad (5a)$$

$$-[M(\xi]\dot{u}^*+[C(\xi)]u^*+[S]w+[Q_1]v=0 \quad (5b)$$

$$[S]\dot{w}+[K(\xi)]u^*+[Q_0]u=0 \quad (5c)$$

$$[S]\dot{u}-[S]v=0 \quad (5d)$$

$$[M(\xi)]\dot{v}+[C(\xi)]v+[K(\xi)]u=[G(\xi)]f \quad (5e)$$

The symmetry of $[M]$, $[C]$, $[K]$, and $[S]$ has been used in deriving Eqs. (5a)-(5e). Transversality conditions corresponding to a free final time as well as those corresponding to any free components of the initial position and final velocity are also obtained easily.[17,19]

Necessary Conditions for Optimal Parameters

In addition to Eqs. (5a-e), it is necessary for the structural parameters to satisfy

$$\sum_{i=1}^{N_d} \delta\xi_i \left\{ \alpha \frac{\partial P}{\partial \xi_i} + \int_0^{t_f} u^{*T} \left(\left[\frac{\partial M}{\partial \xi_i}\right] \dot{v} + \left[\frac{\partial C}{\partial \xi_i}\right] v + \left[\frac{\partial K}{\partial \xi_i}\right] u - \left[\frac{\partial G}{\partial \xi_i}\right] f \right) \mathrm{d}t \right\} = 0 \tag{5f}$$

where $\delta\xi_i$ is the variation of ξ_i. When an optimal structural parameter is on the boundary of the admissible parameters, $\xi_i = 0$ must be used. In the neighborhood of extrema not on the boundary, however, the coefficients of each variation must vanish. The resulting N_d necessary conditions to be satisfied in addition to Eqs. (5a)-(5e) are

$$\alpha \frac{\partial P}{\partial \xi_i} + \int_0^{t_f} u^{*T} \left(\left[\frac{\partial M}{\partial \xi_i}\right] \dot{v} + \left[\frac{\partial C}{\partial \xi_i}\right] v + \left[\frac{\partial K}{\partial \xi_i}\right] u - \left[\frac{\partial G}{\partial \xi_i}\right] f \right) \mathrm{d}t = 0, \quad i = 1,2,\ldots,N_d \tag{6}$$

Numerical Solution for an Optimum

The necessary conditions (5) are a hybrid system of coupled nonlinear equations. Because there is no hope for an analytical solution, a numerical solution is necessary. Many different approaches for numerically solving Eqs. (5) are possible and none in particular is advocated here. Instead, the method used for the numerical examples of this paper is presented in the Appendix.

It is important to point out that solving for the optimal values of a large number of stuctural parameters, the situation likely to occur when an actual complex structure is considered, is plagued with difficulties. First, for a complex structure the dimension N of the equations of motion (1) can be large. A direct approach to the solution of the necessary conditions requires that a two point boundary value problem in time be solved at every iteration of an iterative algorithm for updating the structural parameters (see the Appendix). Solving the two-point boundary value problem accurately is possible only for relatively low dimension systems (e.g., $N<100$). Moreover, its solution can be expensive to obtain, so that iteratively solving for the structural parameters might be prohibitively expensive. Second, as already discussed, the cost functional can be a complicated function of the parameters which is only locally convex. Therefore, an initial guess for the structural parameters in the neighborhood of the optimum is required. In general, such an initial guess can be difficult to come by. It is quite possible that an initial guess far from the optimum is in a region for which the linear equations of motion are not valid, or for which the two-point boundary value problem is exceedingly difficult to solve accurately. In addition, it should be kept in mind that the mathematical model discussed here is only a discretization of an actual system of partial differential equations, and any chosen discretization may not be appropriate for certain values of the structural parameters.

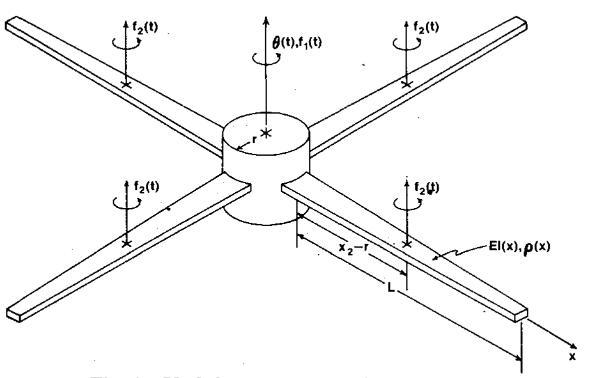

Fig. 1 Undeformed symmetric structure.

Solutions to these problems are beyond the scope of this paper, although they should not be insurmountable with careful attention. For example, the Rayleigh-Ritz method can be used to reduce the number of degrees of freedom so that the two-point boundary value problem is tractable even for complex structures. To this end, a small number of actual eigenvectors are perhaps sufficient to approximate a structure's motion for the purpose of design. On the other hand, to produce good initial guesses to the structural parameters, a multilevel series of approximations to the desired vector of parameters can perhaps be exploited.

Single-Axis Slew Maneuvers of a Symmetric Flexible Structure

An idealized four-boom structure (Fig. 1) has served as the basis for many illustrative examples in the literature[11,13,14] and it is used again here. The booms are of length L, and they are fixed to a rigid hub with radius r and moment of inertial I_h about the axis of rotation. They are capable of bending displacements in the plane of rotation which are assumed to be the same for each boom and to be antisymmetric. Unlike Refs. 11, 13, and 14, however, here the mass per unit of length $\rho(x)$ and the bending stiffness $EI(x)$ of the booms are free to be designed. They will be written as $\rho(x,\xi)$ and $EI(x,\xi)$ to show the explicit dependence on ξ. A control torque $f_1(t)$ acts on the rigid hub and additional idealized discrete control torques $f_2(t),\ldots,f_{N_c}(t)$ can be applied on each boom at distances $x_2,\ldots,x_{N_c}$, respectively, from the center of rotation (only one torque $f_2(t)$ is shown in Fig. 1).

Equations of Motion

Linearized equations of motion governing the single rotation and the antisymmetric in-plane elastic deformations of the structure are presented in Refs. 13 and 14. The equations are valid for small elastic deformations about a trivial equilibrium and they have the form of Eq. (1) with $[C]=[0]$. The bending displacement for each boom is spatially discretized, i.e., represented as the sum of N_μ admissible functions $\phi_i(x-r)$ $(r \leq x \leq r+L)$ multiplying the time dependent generalized coordinates $\mu_i(t)$ $(i=1,2,\ldots,N_\mu)$. To be admissible, the functions $\phi_i(x-r)$ must be twice differentiable and satisfy the geometric boundary conditions $\phi_i(0)=\phi_i'(0)=0$, where in the present context $(\;)'$ denotes the derivative with respect to x. If the angle of rotation is denoted by $\theta(t)$, the generalized coordinate vector $u(t)$ in Eq. (1) is $u=\{\theta,\mu_1,\mu_2\ldots,\mu_{N_\mu}\}^T$; its dimension is $N=N_\mu+1$. Therefore, the coefficient matrices have the partitioned form

$$[M(\xi)] = \begin{bmatrix} I(\xi) & M^T_{\theta\mu}(\xi) \\ M_{\theta\mu}(\xi) & [M_{\mu\mu}(\xi)] \end{bmatrix} \tag{7a}$$

$$[K(\xi)] = \begin{bmatrix} 0 & 0^T \\ 0 & [K_{\mu\mu}(\xi)] \end{bmatrix} \tag{7b}$$

$$[G(\xi)] = [G] = \begin{bmatrix} 1 & 4 & \ldots & 4 \\ 0 & 4\phi_1'(x_2-r) & \ldots & 4\phi_1'(x_{N_c}-r) \\ 0 & 4\phi_2'(x_2-r) & \ldots & 4\phi_2'(x_{N_c}-r) \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 4\phi N_\mu'(x_2-r) & \ldots & 4\phi_{N_\mu'}(x_{N_c}-r) \end{bmatrix} \tag{7c}$$

where the moment of inertia of the undeformed structure about the axis of rotation is

$$I(\xi) = I_h + 4\int_r^{r+L} \rho(x,\xi)x^2 \mathrm{d}x \tag{8a}$$

the ith entry of $M_{\theta\mu}(\xi)$ is

$$M_{\theta\mu,i}(\xi) = 4\int_r^{r+L} \rho(x,\xi)\phi_i(x-r)x\mathrm{d}x, \quad i=1,2,\ldots,N_\mu, \tag{8b}$$

and the ijth entries of $[M_{\mu\mu}(\xi)]$ and $[K_{\mu\mu}(\xi)]$ are

$$M_{\mu\mu,ij}(\xi) = 4\int_r^{r+L} \rho(x,\xi)\phi_i(x-r)\phi_j(x-r)\mathrm{d}x, \quad i,j=1,2,\ldots,N_\mu \tag{8c}$$

$$K_{\mu\mu,ij}(\xi) = 4\int_r^{r+L} EI(x,\xi)\phi_i''(x-r)\phi_j''(x-r)\mathrm{d}x, \quad i,j=1,2,\ldots,N_\mu \tag{8d}$$

Here, the explicit dependence of $\rho(x,\xi)$ and $EI(x,\xi)$ on the entries $\xi_i(i=1,2,\ldots,N_d)$ of ξ must also be considered. Note that ρ and EI are at least nonnegative piecewise continuous functions of x. Rather than maintaining complete generality, it is convenient to assume that each boom's cross section is rectangular with a given height and a width $w(x,\xi)$ that is piecewise continuous, that the booms are solid, and that each boom is made of the same homogeneous material. The width $w(x,\xi)$ is, of course, nonnegative. Moreover, assume that it is essential in order to meet mission requirements for each boom to be at least a thin wire, i.e., to have at least the small uniform mass per unit length ρ_0. The stiffness of the wire is assumed to be negligible for simplicity. Therefore,

$$\rho(x,\xi) = \rho_0 + \rho_a \hat{w}(x,\xi) \tag{9a}$$

$$EI(x,\xi) = EI_a \hat{w}^3(x,\xi) \tag{9b}$$

where ρ_a and EI_a are constants reflecting the specific material and $\hat{w}$ is the dimensionless width (w divided by a characteristic length) of the structure added to the wire. The width can be represented as the sum of N_d functions $\psi_i(x-r)$ multiplying the parameters ξ_i

$$\hat{w}(x,\xi) = \sum_{i=1}^{N_d} \psi_i(x-r)\xi_i \tag{10}$$

Nondimensional Problem Formulation

Next, a reformulation in terms of dimensionless parameters is desirable. The structural design parameters are already dimensionless because $\hat{w}$ is dimensionless. The time t can be normalized by dividing by the final time, i.e., by introducing the dimensionless time $\hat{t}(0 \le t \le 1)$, where $t = t_f \hat{t}$. In addition, ρ_a can be taken as a characteristic mass per unit of length and L can be taken as a characteristic length. Dimensionless admissible functions $\hat{\phi}_i = \phi_i/L$, a length $\hat{x}$, where $x = L\hat{x} + r$, and a vector of generalized forces $\hat{f} = t_f^2 f/\rho_a L^3$ can then be introduced. The introduction of dimensionless admissible functions $\hat{\phi}_i$ renders the components of u dimensionless.

In the sequel, dimensionless versions of Eqs. (1) and (2) are considered for the four-boom structure. They have the form

$$[\hat{M}(\xi)]u^{**}(\hat{t}) + [\hat{K}(\xi)]u(\hat{t}) = [\hat{G}]\hat{f}(\hat{t}) \tag{11}$$

$$J(\hat{f},\xi) = \alpha\hat{P}(\xi) + \int_0^1 \frac{1}{2}(\hat{f}^T[\hat{R}]\hat{f} + \hat{v}^T[\hat{Q}_1]\hat{v} + u^T[\hat{Q}_0]u)\mathrm{d}\hat{t} \tag{12}$$

where $(\ )^*$ denotes differentiation with respect to $\hat{t}$ and $\hat{v} = u^* = t_f\dot{u}$. $[\hat{M}]$ and $[\hat{K}]$ have a form similar to Eqs. (7a) and (7b) except that they are in terms of

$$\hat{I}(\xi) = \hat{I}_h + 4\int_0^1 (\hat{\rho} + \hat{w}(\hat{x},\xi))(\hat{x}+\hat{r})^2\mathrm{d}\hat{x} \tag{13a}$$

$$\hat{M}_{\theta\mu,i}(\xi) = 4\int_0^1 (\hat{\rho} + \hat{w}(\hat{x},\xi))(\hat{x}+\hat{r})\hat{\phi}_i(\hat{x})\mathrm{d}\hat{x}, \quad i=1,2,\ldots,N_\mu, \tag{13b}$$

$$\hat{M}_{\mu\mu,ij}(\xi) = 4\int_0^1 (\hat{\rho} + \hat{w}(\hat{x},\xi))\hat{\phi}_i(\hat{x})\hat{\phi}_j(\hat{x})\mathrm{d},\hat{x} \quad i,j=1,2,\ldots,N_\mu \tag{13c}$$

$$\hat{K}_{\mu\mu,ij}(\xi) = 4\int_0^1 \hat{E}I\hat{w}^3(\hat{x},\xi)\hat{\phi}_i''(\hat{x})\hat{\phi}_j''(\hat{x})\mathrm{d}x, \quad i,j=1,2,\ldots,N_\mu \tag{13d}$$

where the dimensionless parameters

$$\hat{\rho} = \frac{\rho_0}{\rho_a}, \quad \hat{r} = \frac{r}{L}, \quad \hat{I}_h = \frac{I_h}{\rho_a L^3}, \quad \hat{E}I = \frac{t_f^2 EI_a}{\rho_a L^4} \tag{14a-d}$$

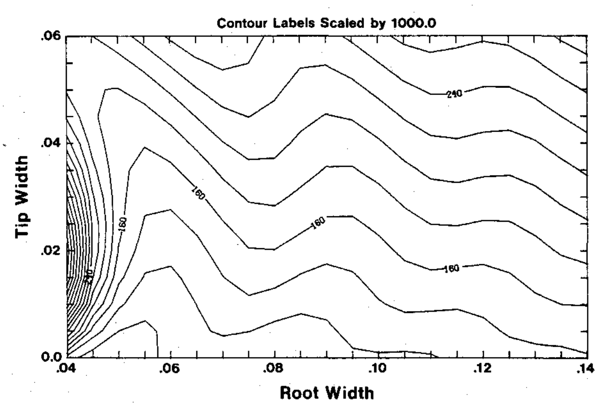

Fig. 2 Rest-to-rest cost vs parameters ($N_\mu = N_c = 1$, $\alpha = 0$).

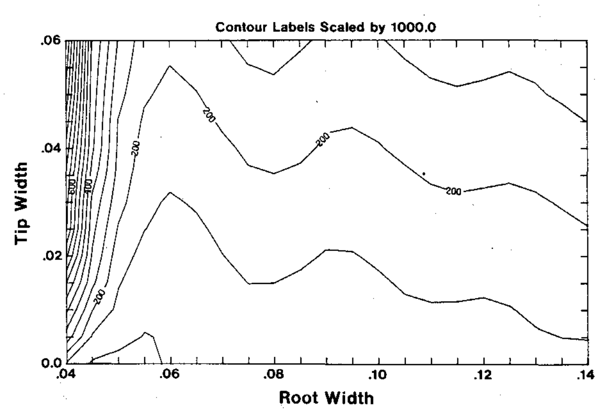

Fig. 3 Rest-to-rest cost vs parameters ($N_\mu = 4$, $N_c = 1$, $\alpha = 0$).

have been used. Moreover, $[\hat{G}]$ is similar to Eq. (7c) except that it is in terms of $\hat{\phi}_i$ and the points $\hat{x}_i = x_i/L$ $(i=2,\ldots,N_c)$, and in the present context $(\ )'$ denotes differentiation with respect to $\hat{x}$.

Examples

The examples presented here all use the specific dimenionless quantities

$$\hat{\rho}=10^{-8}, \quad \hat{r}=0.025, \quad \hat{I}_h=0.1, \quad \hat{EI}=1000.0$$

$$\hat{\phi}_i(\hat{x}) = \hat{x}^{i+1}, \quad i=1,2,\ldots,N_\mu$$

$$\alpha\hat{P}(\xi) = 4\alpha \sum_{i=1}^{N_d} \left(\xi_i \int_0^1 \hat{\psi}_i(\hat{x})\,\mathrm{d}\hat{x}\right)$$

$$[\hat{R}] = R[1], \quad [\hat{Q}_1] = [0], \quad [\hat{Q}_0] = Q_0 \begin{bmatrix} 0 & 0^T \\ 0 & [\hat{Q}_{0\mu\mu}] \end{bmatrix}$$

where $[1]$ is the identity matrix, and the ijth entry of $[\hat{Q}_{0\mu\mu}]$ is

$$\hat{Q}_{0\mu\mu,ij} = \int_0^1 \hat{\phi}_i(\hat{x})\,\hat{\phi}_j(\hat{x})\,d\hat{x}, \qquad i,j=1,2,\ldots,N_\mu \tag{15}$$

The magnitudes R and Q_0 are taken to be unity. Note that the weighting matrix $[\hat{Q}_0]$ is chosen so as to penalize the square integral norm of the booms' elastic deformations.

It is illustrative to consider a problem that has only two structural parameters, ξ_1 and ξ_2 ($N_d=2$). For any specified ξ_1 and ξ_2, an optimal control and a cost J can be calculated. The cost is then plotted vs ξ_1 and ξ_2 to illustrate the complexity of its behavior (here contour plots will be given). In practice, a nonlinear programming procedure must be used to find a global minimum of the cost when there are more than two parameters.

A specific two-parameter problem is defined for this paper by choosing two basis functions $\hat{\psi}_1$ and $\hat{\psi}_2$ in the representation (10) of the booms' widths. For simplicity, the structure added to the thin wire is assumed to be linearly tapered as shown in Fig. 1. In this case,

$$\hat{\psi}_1(\hat{x}) = 1-\hat{x} \tag{16a}$$

$$\hat{\psi}_2(\hat{x}) = \hat{x} \tag{16b}$$

so that

$$\xi_1 = \hat{w}(0) = \text{root width}, \qquad \xi_2 = \hat{w}(1) = \text{tip width}$$

Two dimensionless maneuvers are considered. The first is a rest-to-rest maneuver from $u(0)=\hat{v}(0)=\mathbf{0}$ to $\hat{v}(1)=\mathbf{0}$, $u(1)=\{1,0,\ldots,0\}^T$, i.e., $\theta(1)=1$. The second maneuver is a spinup from $u(0)=\hat{v}(0)=\mathbf{0}$ to $\hat{v}(1)=\{1,0,\ldots,0\}^T$ $u(1)=\{1,0,\ldots,0\}^T$.

Rest-to-Rest Maneuver

$$N_c=1, \qquad \alpha=0$$

Figures 2 and 3 are contour plots of maneuver cost vs the design parameters ξ_1 and ξ_2 for one ($N_\mu=1$) and four ($N_\mu=4$) admissible functions, respectively. The maneuver is performed using only the hub torque and the mass penalty is taken to be zero. In each case, the optimal value of tip width ξ_2 is zero, which is an obvious result. The optimal value of root width ξ_1 changes only slightly from $\xi_1=0.0477$ to $\xi_1=0.0474$ as the number of admissible functions is increased. Time histories evaluated at the respective optimal values of the parameters are shown in Figs. 4 and 5.

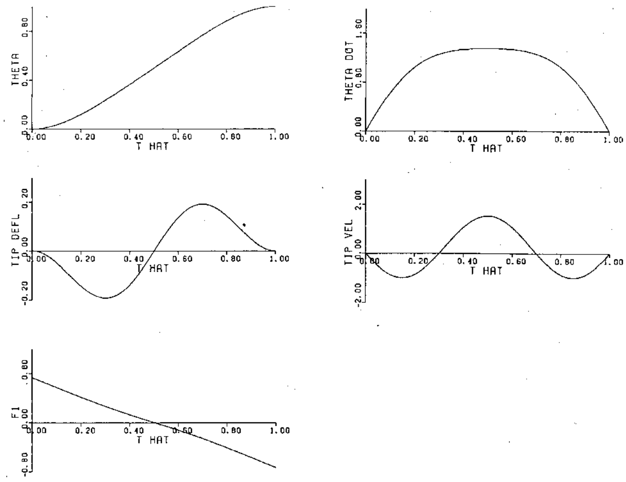

Fig. 4 Rest-to-rest time histories for the optimal structure ($\xi_1=0.0477$, $\xi_2=0$, $N_\mu=N_c=1$, $\alpha=0$).

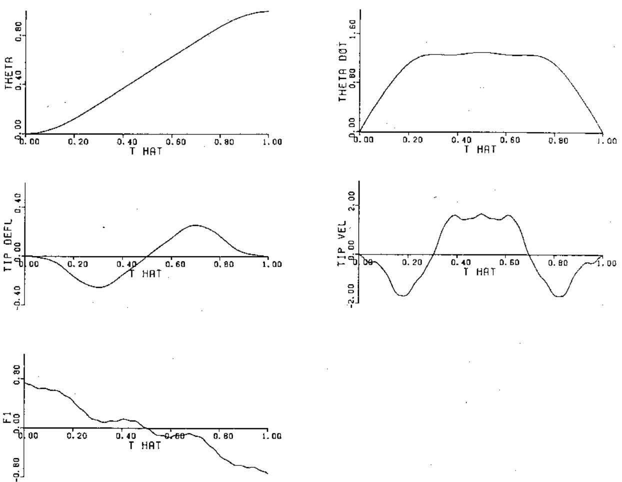

Fig. 5 Rest-to-rest time histories for the optimal structure ($\xi_1=0.0474$, $\xi_2=0$, $N_\mu=4$, $N_c=1$, $\alpha=0$).

Note that the complexity of the contour plots changes significantly, even for this simple problem. In theory the error in calculating the cost should decrease as N_μ is increased. Numerical difficulties are encountered, however, as N_μ is increased. The difficulties are attributed to the presence of both low- and increasingly high-frequency modeled dynamics. Numerical analysts refer to such a system of equations as being *stiff*. The error was found actually to increase rapidly when more than four or five admissible functions are used. Because the main concern here is with small values of N_μ, no attempt was made to overcome this problem. In addition, at the beginning where to look in the ξ_1, ξ_2 plane for a minimum is not known and numerical values of the cost should be interpreted cautiously at points removed from the minimum. For example, at values of root width that are smaller than 0.04, the booms are quite flexible and their deflections during the maneuver can be sufficiently large that the linear equations are not valid. This is particularly true at values of tip width greater than the root width and is seen in the time histories given in Fig. 6.

$$N_\mu=1, \qquad N_c=1, \qquad \alpha=1$$

Figure 7 is a contour plot of cost vs the design parameters for $N_\mu=1$ and a nonzero mass penalty. The maneuver is performed using only the hub torque. Note that the costs are greater than in Fig. 1 and the minimum is more pronounced. At the minimum, the optimal value of root width is $\xi_1 = 0.0448$. As expected, the mass penalty dominates at points far from the origin so that the cost contours approach straight lines there.

$$N_\mu = 4, \qquad N_c = 2, \qquad \alpha = 0$$

Figure 8 is a contour plot of maneuver cost vs the design parameters for $N_\mu = 4$, no penalty on the mass, and five torques ($N_c = 2$). The weighting matrix $[R]$ is the identity matrix so that the boom torques are penalized less than the hub torque, and there is a unit penalty on the sum of the four boom torques. Consequently, the boom torques contribute substantially to the maneuver and the optimum structure is significantly different. The effect is that stiffer booms are optimal, where the optimal value of the root width is $\xi_1 = 0.115$. The behavior of the cost contours is similar to previous figures but at higher values of root width. Time

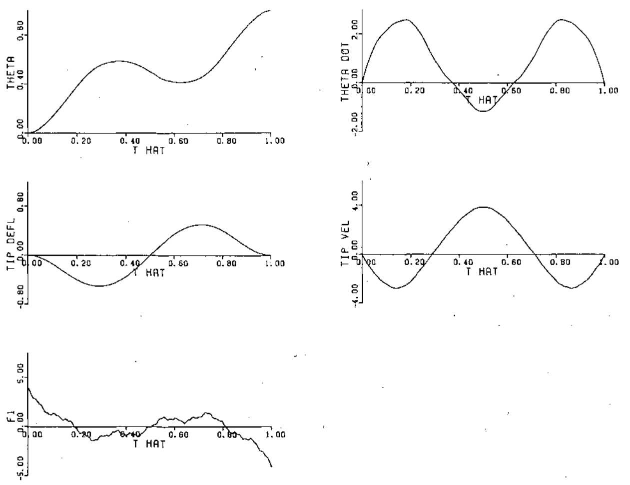

Fig. 6 Rest-to-rest time histories for a nonoptimal structure ($\xi_1 = 0.04$, $\xi_2 = 0.06$, $N_\mu = 4$, $N_c = 1$, $\alpha = 0$).

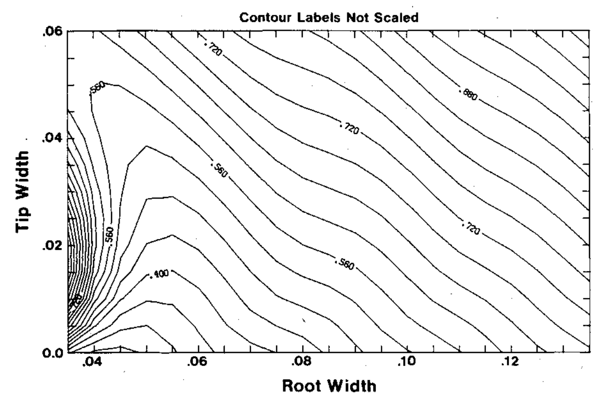

Fig. 7 Rest-to-rest cost vs parameters ($N_\mu = N_c = 1$, $\alpha = 1$).

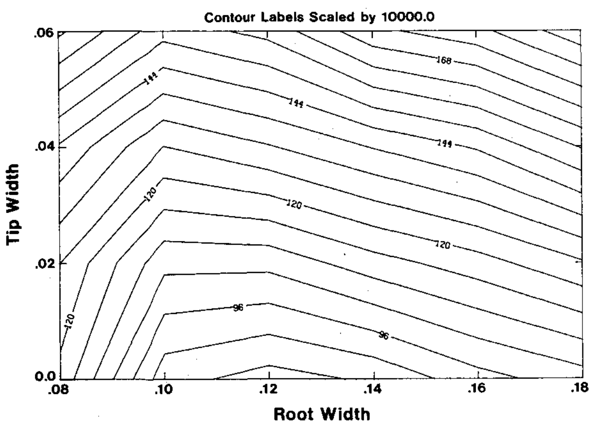

Fig. 8 Rest-to-rest cost vs parameters ($N_\mu = 4$, $N_c = 2$, $\alpha = 0$).

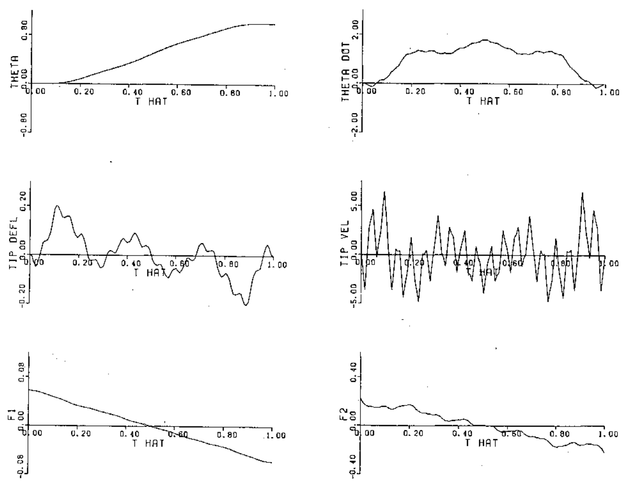

Fig. 9 Rest-to-rest time histories for the optimal structure ($\xi_1 = 0.115$, $\xi_2 = 0$, $N_\mu = 4$, $N_c = 2$, $\alpha = 0$).

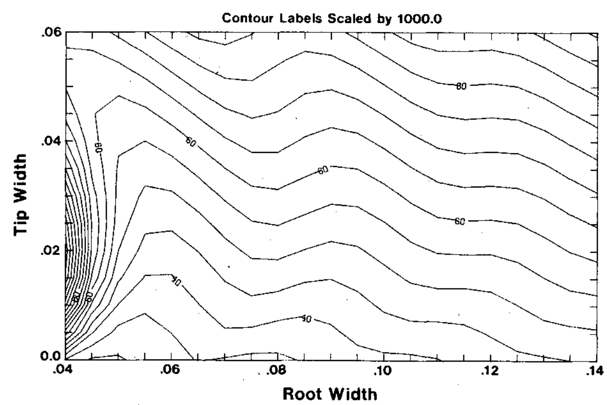

Fig. 10 Spinup cost vs parameters ($N_\mu = 4$, $N_c = 1$, $\alpha = 0$).

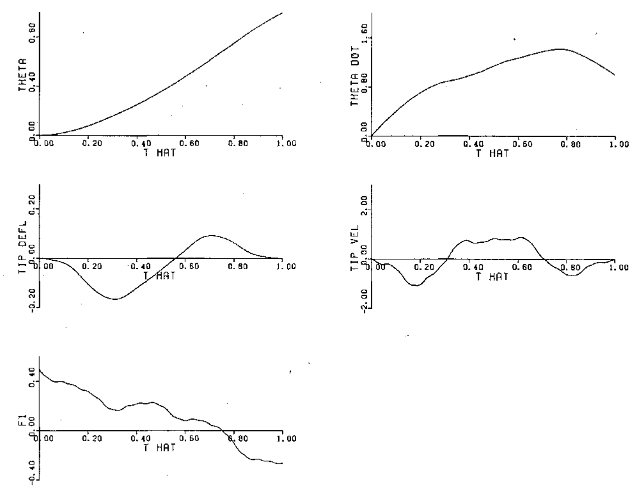

Fig. 11 Spinup time histories for the optimal structure ($\xi_1 = 0.0472$, $\xi_2 = 0$, $N_\mu = 4$, $N_c = 1$, $\alpha = 0$).

histories evaluated at the optimal parameters are shown in Fig. 9. Note that the hub torque is small relative to the boom torques. Also, the torques have substantial high frequency components.

Spinup Maneuver

$$N_c = 1, \qquad \alpha = 0$$

Figure 10 is a contour plot of maneuver cost vs the deisgn parameters for $N_\mu = 1$, only a hub torque, and no penalty on the mass. Although the maneuver is different, the optimal structure is nearly the same as for a rest-to-rest maneuver (here $\xi_1 = 0.0472$ is optimal). Time histories evaluated at the optimal parameters when $N_\mu = 4$ are shown in Fig. 11.

Concluding Remarks

The structural design and the control design problems are unified mathematically in this paper to create a single optimization problem. For simplicity, only open-loop maneuvers of flexible structures from a specified initial state to a specified final state in a specified time interval have been considered. The ideas are extendable to problems with a free final time, and to those with a free final state that is penalized quadratically in the cost functional. The paper also forms a foundation for considering other, more complex problems, such as those containing bounded control forces, bounded states, and/or more complex structural parameter constraints. It would also be of interest to solve for an optimal structure and its control when the initial and final states are contained in known sets, but are otherwise free. Such problems are in some ways analogous to structural optimization for multiple loads. Another related problem would be to solve for an optimal structure in the presence of quantified uncertainties in some structural parameters, leading to a robust design. In fact, as a reviewer pointed out, "this is particularly important here, as a large structure may have to support different payload masses at different times."

Numerical examples have been presented for single-axis slew maneuvers of a symmetric structure having four identical flexible booms. Extremal solutions depend not only on the weighting matrices chosen in the cost functional, but on the number of admissible functions used to represent the elastic deformation as well. Care must be used to ensure that the deformation can be represented accurately for a range of values of the structural design parameters. Errors can arise because of 1) using too few admissible functions, 2) numerical errors associated with calculating the cost, and 3) large deflections so that the linear equations are not valid. Finally, the inclusion of a weighted mass term in the cost functional conditions the functional so that its global minimum is more pronounced and an extremal control-parameter pair can be found more easily.

Appendix
Numerical Solution of the Necessary Conditions

It is first convenient to write Eqs. (5a-e) more compactly. The dispacement and velocity vectors form a $2N$ state vector $y = \{u^T, v^T\}^T$, and the adjoint displacement and Lagrange multiplier vectors form a 2 N costate vector $y^* = \{w^T, u^{*T}\}^T$. Upon introducing the $4N$ vector $z = \{y^T, y^{*T}\}^T$, Eqs. (5b)-(5e) after substitution of Eq. (5a) are symbolically

$$[A(\xi)]\dot{z} + [B(\xi)]z = 0 \tag{A1}$$

where the $4N \times 4N$ matrices $[A(\xi)]$ and $[B(\xi)]$ are

$$[A(\xi)] = \begin{bmatrix} [S] & [0] & [0] & [0] \\ [0] & [M(\xi)] & [0] & [0] \\ [0] & [0] & [S] & [0] \\ [0] & [0] & [0] & [M(\xi)] \end{bmatrix} \tag{A2a}$$

$$[B(\xi)] = \begin{bmatrix} [0] & -[S] & [0] & [0] \\ [K(\xi)] & [C(\xi)] & [0] & -[G(\xi)][R]^{-1}[G(\xi)]^T \\ [Q_0] & [0] & [0] & [K(\xi)] \\ [0] & -[Q_1] & -[S] & -[C(\xi)] \end{bmatrix} \tag{A2b}$$

For a specific vector of parameters ξ^k, an optimal control force, denoted by f^k, is found from Eq. (5a) after the optimal trajectory $z^k(t)$ for $[0, t_f]$ is obtained by solving Eq. (A1). The optimal control always exists if the structure is completely controllable from the N_c entries of the force vector f. It can also exist when the structure is not completely controllable but the initial state is in the set of controllable states and the final state is in the set of reachable states. Of course, controllability can depend on ξ^k, and when complete controllability is missing numerical difficulties arise.

Solution for Initial Costates—Final States Given

Equation (A1) governs the two-point boundary value problem with a specified initial state $y(0)$ and the final state $y(t_f) \in Y$. When all components of the final state are known and $|A(\xi^k)]$ is nonsingular, the unknown initial costate vector $y^*(0)$ can be found directly from the fundamental solution to Eq. (A1),[14,19] i.e., from

$$z^k(t_f) = [\Phi^k(t_f)]z^k(0) \tag{A3}$$

where $[\Phi^k(t_f)]$ is the state transition matrix for the system. Computing the transition matrix accurately can be difficult, especially for large values of t_f and/or for values of the structural parameters that render Eq. (A1) very stiff in the numerical anlaysis sense. Moreover, numerical problems occur when $[A(\xi^k)]$ is singular, i.e., when $[M(\xi^k)]$ is singular, as can happen in certain special cases in some problems. Ignoring these potential difficulties, Eq. (A3) can be conformally partitioned as

$$\begin{Bmatrix} y(t_f) \\ y^{*k}(t_f) \end{Bmatrix} = \begin{bmatrix} [\phi^k_{yy}(t_f)] & [\phi^k_{y^*}(t_f)] \\ [\phi^k_{*y}(t_f)] & [\phi^k_{*y}(t_f)] \end{bmatrix} \begin{Bmatrix} y(0) \\ y^{*k}(0) \end{Bmatrix} \tag{A4}$$

so that the initial costate vector is the solution to the $2N$ equations

$$[\phi^k_{y^*}(t_f)]y^{*k}(0) = y(t_f) - [\phi^k_{yy}(t_f)]y(0)] \tag{A5}$$

Solution for the Trajectory and Cost

A discrete time history of $z^k(t)$ must also be computed, perhaps by dividing $[0, t_f]$ into $2N_t$ equal intervals Δt, and computing

$$z^k(j\Delta t) = z^k_j = [\Phi^k(\Delta t)]z^k_{j-1}, \qquad j = 1, 2, \ldots, 2N_t \tag{A6}$$

Then, the total cost $J^k = J(f^k, \xi^k)$ for the maneuver can be approximated as a discrete sum by using Simpson's rule.

Updating the Structural Parameters

The N_d equations (6) can be collected into a vector $L(\xi)$ that depends implicitly on ξ through the solution to Eq. (A1). The vector $L(\xi)$ is the gradient of the cost functional with respect to the structural parameters. A parameter updating procedure is based on the gradient. By solving Eq. (5e) for $\dot{v}$ and using Eq. (5a) to replace f, each component of the gradient can be written in the form

$$L_i(\xi) = \alpha \frac{\partial P}{\partial \xi_i} + \int_0^{t_f} u^{*T} \left\{ \left(\left[\frac{\partial C}{\partial \xi_i} \right] - \left[\frac{\partial M}{\partial \xi_i} \right] [M(\xi)]^{-1} [C(\xi)] \right) v + \left(\left[\frac{\partial K}{\partial \xi_i} \right] - \left[\frac{\partial M}{\partial \xi_i} \right] [M(\xi)]^{-1} [K(\xi)] \right) u - \left(\left[\frac{\partial G}{\partial \xi_i} \right] - \left[\frac{\partial M}{\partial \xi_i} \right] [M(\xi)]^{-1} [G(\xi)] \right) [R]^{-1} [G(\xi)]^T u^* \right\} \mathrm{d}t \tag{A7}$$

By using the discrete values of the trajectory and Simpson's rule, each component $L_i^k = L_i(\xi^k)$ of the gradient can also be calculated approximately.

Starting from an initial choice of structural parameters $\xi^0 > \mathbf{0}$, and the control force f^0 and cost J^O, it is desired to move iteratively toward the optimal solution ξ^O, f^O, and minimum cost $J^O = J(f^O, \xi^O)$. At any iteration k, there are two questions; what direction to move, and how far? The general updating formula for the nonlinear program is[20,21]

$$\xi^{k+1} = \xi^k + \beta^k \eta^{k+1}, \qquad k = 0, 1, \ldots \tag{A8}$$

The direction η^{k+1} can be chosen to depend on the gradient of the cost functional via

$$\eta^{k+1} = -[H^k] L(\xi^k) \tag{A9}$$

where $[H^k]$ is an $N_d \times N_d$ matrix. A suitable algorithm for choosing $[H^k]$ and β^k is the gradient projection algorithm developed by Goldfarb[21] for linear inequality constraints, here Eqs. (3), and based on symmetric quasiNewton (e.g., Davidon-Fletcher-Powell[22]) updates of $[H^k]$. Convergence is obtained when Eqs. (3) are satisfied and simultaneously, the projection of η^k onto a subspace orthogonal to all the active constraints is smaller in norm than a small positive number.

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